ON PERIODIC MOTIONS OF A CLASS OF AUTONOMOUS SYSTEMS

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In [1,2] a method is given for finding the periodic motions of nonlinear systems described by equations (1.1) of a particular form. This method is used in the present paper to prove a theorem on the existence of periodic solutions of equations (1.1) under more general conditions than in Dragilev's theorem [3] on the existence of at least one limit cycle. The theorems proved earlier made it possible to justify the formulation and solution of intermediate and mixed problems for the study of nonlinear systems described by equations of the form (1.1), and to prove that the results obtained may also be used to solve the direct problem; that is, to find all the periodic solutions of equation (1.1), at least under the condition that the functions $p(x)$ and $q(x)$ are continuous.

1. A prominent class among autonomous systems with one degree of freedom is the class of systems described by equations of the form

$$x^{\cdot\cdot} + p(x)\, x^{\cdot} + xq(x) = 0 \tag{1.1}$$

Here $p(x)$ and $q(x)$ are real functions of x. We introduce the functions $\Phi(x)$ and $\Psi(x)$, related to $p(x)$ and $q(x)$ by the system of equations

$$\frac{1}{2}(1 - \Phi\Psi^2)\frac{d\Phi}{dx} + xq(x) = 0, \qquad \frac{3}{2}\Psi\frac{d\Phi}{dx} + \Phi\frac{d\Psi}{dx} + p(x) = 0 \tag{1.2}$$

Theorem 1.1. Let one of the particular solutions of system of equations (1.2) be expressible by two real functions $\Phi = \varphi(x)$ and $\Psi = \psi(x)$, which in some interval $[a \leqslant x \leqslant b]$, $a \neq b$, satisfy the conditions

$$\varphi = 0, \quad d\varphi / dx \neq 0 \quad \text{if} \quad x = a_1, b \tag{1.3}$$

φ is continuous in $[a, b]$; $0 < \varphi$ in (a, b)

$$\sqrt{\varphi}|\psi| < 1 \quad \text{in } [a, b] \tag{1.4}$$

Then to the above two functions $\varphi(x)$ and $\psi(x)$ there corresponds a particular periodic solution $x(t + t_0)$ of equation (1.1) with extremum values at $x = a$, $x = b$ a finite non-zero period

$$T = 2\int_a^b \frac{dx}{\sqrt{\varphi}\,(1 - \varphi\psi^2)} \tag{1.5}$$

Proof. Equation (1.1) has a first integral

$$x^{\cdot} = \pm\sqrt{\Phi}(1 \pm \sqrt{\Phi}\Psi) = \pm\sqrt{\Phi} + \Phi\Psi \tag{1.6}$$

where Φ and Ψ are the general solution of system of equations (1.2). Indeed, if we differentiate (1.6) with respect to the independent variable t, after the proper transformation, we obtain equation (1.1).

According to (1.6), at least under the condition that the two functions φ and ψ defined by (1.2), satisfy conditions (1.3) and (1.4), in the interval $[a, b]$ the phase trajectory of the corresponding solution $x(t + t_0)$ of equation (1.1) describes in the $\dot{x}x$ phase plane a closed curve which does not intersect itself, is nowhere tangent to the x-axis, and intersects it only at the two end points a and b, so that the solution $x(t + t_0)$ is a periodic function with the extremum values a and b and with a bounded non-zero period T. Indeed, from (1.6) we have

$$\begin{aligned} t + t_0 &= \int_a^{x \leqslant b} \frac{d\mu}{\sqrt{\varphi(\mu)}\,(1 + \sqrt{\varphi}\psi)} \qquad \text{if } \frac{dx}{dt} \geqslant 0 \\ t + t_0 &= \int_a^b \frac{dx}{\sqrt{\varphi}\,(1 + \sqrt{\varphi}\psi)} - \int_b^{x \geqslant a} \frac{d\mu}{\sqrt{\varphi}\,(1 - \sqrt{\varphi}\psi)} \qquad \text{if } \frac{dx}{dt} < 0 \end{aligned} \tag{1.7}$$

The period of oscillation is found from the equation

$$T = \int_a^b \frac{dx}{\sqrt{\varphi}\,(1 + \sqrt{\varphi}\psi)} + \int_a^b \frac{dx}{\sqrt{\varphi}\,(1 - \sqrt{\varphi}\psi)} = 2\int_a^b \frac{dx}{\sqrt{\varphi}\,(1 - \varphi\psi^2)}$$

The theorem is proved. It should be noted that there are infinitely many real functions $p(x)$ and $q(x)$ such that the conditions of Theorem 1.1 are satisfied but the conditions of Dragilev's theorem [3] are not.

Example. Let $u(x)$ and $v(x)$ be integral polynomials in x which in some interval $[a, b]$ satisfy the conditions

$$u^2 < 1, \qquad \varepsilon^2 v^2 (1 + u^2) [A^2 - (x - c)^2]^{1-\gamma-2\lambda} < 1 \tag{1.8}$$

where ε, γ and λ are positive constants

$$\varepsilon > 0, \qquad 0 \leqslant \gamma < 0.5, \qquad 0 < \lambda \leqslant {}^1/_2 (1 - \gamma) \tag{1.9}$$
$$A = {}^1/_2 (b - a), \qquad c = {}^1/_2 (b - a), \qquad A^2 - (x - c)^2 = (b - x)(x - a)$$

The set of functions $p(x)$ and $q(x)$ is determined from the equations

$$p(x) = \varepsilon^2 (1 + u) \{(3 - 3\gamma - 2\lambda)(1 + u) \ (x - c) v - [3u'v - v'(1 + u)](b - x) \times$$
$$\times (x - a)\} [A^2 - (x - c)^2]^{-\gamma-\lambda}$$
$$xq(x) = \varepsilon^2 [(x - c)(1 - \gamma)(1 + u) - u'(b - x)(x - a)] \{1 - \varepsilon^2 v^2 (1 + u)^2 \times$$
$$\times [A^2 - (x - c)^2]^{1-\gamma-2\lambda}\} [A^2 - (x - c)]^{-\gamma} \tag{1.10}$$

Here the prime denotes differentiation with respect to x.

For $\gamma > 0$ the conditions of Dragilev's theorem are not satisfied at the end points of the interval $[a, b]$, but equations of form (1.1) have periodic solutions with the extremum values $x = a, b$ and finite periods, since if we use (1.10), the particular solutions

$$\Phi = \varepsilon^2 (1 + u)^2 [A^2 - (x - c)^2]^{1-\gamma}, \qquad \Psi = v [A^2 - (x - c)^2]^{-\lambda} \tag{1.11}$$

found from (1.2) will, according to (1.8) and (1.9), satisfy the conditions of Theorem 1.1 in the interval $[a, b]$.

The equations of the phase trajectories of such solutions will have the form

$$x^{\cdot} = + \varepsilon (1 + u) \{1 \pm \varepsilon v (1 + u) [A^2 - (x - c)^2]^{1/_2 (1-\gamma-2\lambda)}\} [A^2 - (x - c)^2]^{1/_2 (1-\gamma)} \tag{1.12}$$

in the $xx^{\cdot}$ phase plane.

The periods of oscillation may be found to any desired accuracy from the equations

$$T = \frac{2}{\varepsilon} \int_{-A}^{A} g(x - c) \frac{d(x - c)}{\sqrt{A^2 - (x - c)^2}} \tag{1.13}$$

since, according to (1.9), the functions $g(x - c)$ can be expanded into absolutely and uniformly convergent power series in $(x - c)$. Here

$$g(x - c) = \frac{[A^2 - (x - c)^2]^{1/_2 \gamma}}{(1 + u)\{1 - \varepsilon^2 v^2 (1 + u)^2 [A^2 - (x + c)^2]^{1-\gamma-2\lambda}\}}$$

As the existence criterion for at least one periodic solution of equation (1.1) with specified $p(x)$ and $q(x)$. Dragilev's theorem cannot be compared with Theorem 1.1, since it is much easier to check that the conditions of Dragilev's theorem are satisfied than to check that the conditions of Theorem 1.1 are satisfied.

However, in the study of autonomous nonlinear systems described by equations of the form (1.1), Theorem 1.1 makes it possible under even more general conditions to prove the solution of the intermediate problem, when the equation of a phase trajectory of form (1.6) is given; and, from it, using the system of equations (1.2), one must find the functions $p(x)$ and $xq(x)$ and equation (1.1). In this case, if conditions (1.3) and (1.4) are satisfied, one particular periodic solution of the resulting equation of form (1.1) obviously is easy to find from equations (1.5) and (1.7). It should be noted that, in addition to this particular periodic solution, equation (1.1) may also have other periodic solutions which must be found by using the solution of the direct problem.

2. In real self-oscillatory systems described by the equations of form (1.1), the functions $p(x)$ and $q(x)$ usually satisfy the conditions

$$p(x), \quad xq(x) \quad \text{continuous}, \qquad q(x) > 0, \quad p(x) \neq 0 \tag{2.1}$$

Hereafter, we shall assume that these conditions are satisfied. In order to study such systems we shall introduce the auxiliary functions $F(x)$ and $H(x)$, related to the first derivatives $x^{\cdot}$ by the equalities

$$\sqrt{F(x)} = x^{\cdot} \quad \text{if} \quad x^{\cdot} \geqslant 0, \qquad \sqrt{H(x)} = -x^{\cdot} \quad \text{if} \quad x^{\cdot} \leqslant 0 \tag{2.2}$$

Hence we have

$$x^{\cdot\cdot} = dF/dx \quad \text{if} \quad x^{\cdot} \geqslant 0, \qquad x^{\cdot\cdot} = dH/dx \quad \text{if} \quad x^{\cdot} < 0 \tag{2.3}$$

Using (2.2) and (2.3), instead of equation (1.1), we obtain two equations

$$dF/dx = -2xq(x) - 2\sqrt{F(x)}\,p(x), \quad dH/dx = 2xq(x) + 2\sqrt{H(x)}\,p(x) \tag{2.4}$$

All the real solutions of equations (2.4) are positive; according to (2.1), the right-hand sides of these equations are continuous and, for $F(x)$ and $H(x) > 0$, satisfy a Lipschitz condition. Therefore, in accordance with (2.2) and (2.3), when (2.1) is satisfied, to each periodic solution $x(t + t_0)$ with extremum values $x = a, b$ there corresponds a pair of functions $f(x)$ and $h(x)$, found from (2.4), and these functions satisfy the conditions

$$
\begin{aligned}
&f(x),\ h(x),\ df/dx,\ dh/dx \quad \text{continuous in } [a,b] \\
&f(x)>0,\ h(x)>0 \quad \text{in } (a,b) \\
&f(x)=0,\ h(x)=0, \quad df/dx \neq 0, \quad dh/dx \neq 0 \quad \text{for } x=a,b
\end{aligned}
\tag{2.5}
$$

According to (1.5) and (2.2), the functions $\Phi(x)$ and $\Psi(x)$ are found from (1.2) as follows:

$$
\sqrt{\Phi(x)} = \frac{1}{2}\left[\sqrt{F(x)} + \sqrt{H(x)}\right], \qquad \Psi(x) = \frac{\sqrt{F(x)} - \sqrt{H(x)}}{2\left[\sqrt{F(x)} + \sqrt{H(x)}\right]^2} \tag{2.6}
$$

From this and from (2.5) it is readily seen that the following theorems are true.

Theorem 2.1. If (2.1) is satisfied, to every periodic solution $x(t + t_0)$ of equation (1.1) with extremum values $x = a$, b there corresponds a pair of functions $\varphi(x)$ and $\psi(x)$, found from (1.2), and these functions are continuous in $[a, b]$, have continuous first derivatives with respect to x, and satisfy condition (1.3) and (1.4).

Theorem 2.2. To any pair of continuous functions $\varphi(x)$ and $\psi(x)$, found from (1.2), which have continuous first derivatives and satisfy conditions (1.3) and (1.4) in some interval $[a, b]$, there corresponds a continuous periodic solution of equation (1.1) with continuous first and second derivatives with respect to t and extremum values $x = a$, b.

According to Theorem 2.2, if (2.1) is satisfied, the functions φ and ψ of (1.2) and their first derivatives with respect to x, corresponding to the periodic solutions of equation (1.1), may be uniformly approximated by integral polynomials $\varphi^{\circ}(x)$ and $\psi^{\circ}(x)$ in x

$$
\varphi(x) \approx \varphi^{\circ}(x), \qquad \psi(x) \approx \psi^{\circ}(x)
$$

It can be shown [1] that according to (1.3) and (1.4), the polynomials φ° and ψ° necessarily satisfy equations of form (1.11) in which the constants γ and λ are equal to zero, and the polynomials $u(x)$ and $v(x)$, if ε is chosen large enough, satisfy conditions (1.9) in the interval $[a, b]$. Equations (1.10) become

$$
\begin{aligned}
p(x) &= \varepsilon^2 (1+u)\{3(x-c)(1+u)v - [3uv' - v'(1+u)](b-x)(x-a)\} \\
xq(x) &= \varepsilon^2 [(x-c)(1+u) - u'(b-x)(x-a)]\{1 - \varepsilon^2 v^2 (1+u)^2 [A^2 - (x-c)^2]\}
\end{aligned}
\tag{2.7}
$$

The equations of the phase trajectories of the periodic solutions can be reduced to the form

$$
x^{\cdot} = \pm\varepsilon(1+u)\{1 \pm \varepsilon v(1+u)[A^2 - (x-c)^2]^{1/2}\}[A^2 - (x-c)^2]^{-1/2} \tag{2.8}
$$

The periods of oscillation are found from an equation of form (1.13), where the functions $g(x - c)$ are

$$g(x-c) = (1+u)^{-1}\{1-\varepsilon^2 v^2(1+u)^2[A^2-(x-c)^2]\}^{-1}$$

In Section 1 above we justified the intermediate formulation of the problem for the general case, when the functions $p(x)$ and $xq(x)$ may be discontinuous.

Evidently, when the intermediate problem is stated and the functions $p(x)$ and $xq(x)$ are required to be continuous, it is desirable to express the functions $\Phi(x)$ and $\Psi(x)$ in form (1.11), with $\gamma = \lambda = 0$. In this case it is a relatively simple matter both to find the functions $p(x)$ and $q(x)$ from equations of form (1.2) and to find a periodic solution for a given closed-cycle equation.

The intermediate formulation of such a problem may be used, for example, in the design of single-loop generators for electromagnetic oscillations, from the quasilinear to the relaxation type, if these systems are described by equations reducible to the form (1.1), in which the functions $p(x)$ and $xq(x)$, in the great majority of cases, satisfy conditions (2.1).

In certain cases in the investigation of systems described by equations of form (1.1), it is desirable to formulate and solve the mixed problem when the functions $p(x)$ and the functions $\Phi(x)$ from (1.2) are given. A special case of such a problem is considered in [2].

In the mixed problem, after solving system (1.2) for $\Psi(x)$, we find

$$\Psi = \Phi^{-1/2}\int p(x)\sqrt{\Phi}\,dx, \quad 2kq(x) + \left\{1-\left[\frac{1}{\Phi}\int p(x)\sqrt{\Phi}\,dx\right]^2\right\}\Phi' = 0 \qquad (2.9)$$

From the second equation we can readily find $xq(x)$.

Example. Let

$$\begin{aligned} p(x) &= \alpha(1-x^2), \qquad \Phi = a^2(A^2-x^2)\left(1+\frac{2^2x^2}{48}-\frac{\alpha^2}{96}x^4\right) \\ a^2 &= \left(1-\frac{\alpha^2A^2}{24}\right)^{-1}, \qquad A^2 = 4\left(1+\frac{\alpha^2}{96}+\frac{7\alpha^4}{8\cdot 24\cdot 96}+\ldots\right)\approx 4\left(1+\frac{\alpha^2}{96}\right) \end{aligned} \qquad (2.10)$$

The equation for a closed cycle is of the form

$$x = \pm a\sqrt{A^2-x^2}\left(1+\frac{\alpha^2x^2}{48}-\frac{\alpha^2x^4}{96}\right)\left[1\pm\frac{\alpha x}{4}\sqrt{A^2-x^2}\left(1-\frac{\alpha^2}{96}+\frac{\alpha^2x^2}{3\cdot 96}-\frac{\alpha^2x^4}{2\cdot 96}\right)\right. \qquad (2.11)$$

Solving an equation of the form (2.9), we find

$$q(x) = a^2[1-D(x)]$$

where $D(x)$ is an even analytic function of x, and

$$D(0) = 0, \qquad |D(x)| < 0.02\,\alpha^4 \quad \text{for } \alpha < 1$$

By analogy with the example given in [2], it may be shown that, at least for $\alpha < 0.5$, an equation of the form

$$x^{\cdot\cdot} + a^2 [1 - D(x)]\, x = \alpha (1 - x^2)\, x^{\cdot}$$

obtained from (2.10), describes, for example, an automatic generator with transformer feedback for soft excitation, with an error of the same order as the Van der Pol equation.

In conclusion, we note that system of equations (1.2) or equation (2.9) may be used to find all the functions $\varphi(x)$ and $\psi(x)$; that is, to solve the direct problem, at least in the cases where the given functions $p(x)$ and $xq(x)$ satisfy conditions (2.1). In this case, according to Theorem 2.1, it can be stated that no periodic solution of equation (1.1) will be lost.

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